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Entropic Repulsion of the Massless Field with a Class of Self-potentials

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Abstract We consider the $d+1$-dimensional effective interface model of gradient type with a quadratic interaction potential and a self-potential. Without the self-potential, the model coincides with the d-dimensional massless Gaussian field. We show that for an arbitrary repulsive self-potential which can be thought as interaction of the interface with a "soft wall", the field is pushed up at least to the same level when the original Gaussian field is conditioned to be positive everywhere, namely the "hard wall" condition is imposed.

Keywords Entropic repulsion · Random interface · Gaussian field · Self-potential

1 Model and Result

Under the situation that two distinct pure phases like crystal/vapor coexist in space, hypersurfaces called interfaces are formed and separate these distinct phases. The effective interface model of gradient type is one of the microscopic modelization of such phase separating interfaces. In this model, the interface in the $d+1$-dimensional space is modeled as the graph of a random height function from $\mathbb{Z}^d$ to $\mathbb{R}$ and its distribution is given by a Gibbs measure with an interaction depending on the gradients of the field. One of the problems related to such interface is the study of localization/delocalization transition by the effect of various external potentials and it has been quite active in recent years. In this paper we consider a class of self-potentials which can be thought as interactions of the interface with a "soft wall" and study its push up effect.

Let $d \geq 2$ and $\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$. For a configuration $\phi = \{\phi_x\}_{x\in\Lambda_N} \in \mathbb{R}^{\Lambda_N} \equiv \Omega_N$, consider the following Hamiltonian:

$$H_N^U(\phi) = H_N(\phi) + \sum_{x\in\Lambda_N} U(\phi_x),$$

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where

$$H_N(\phi) = \frac{1}{8d} \sum_{\substack{\{x,y\}\cap\Lambda_N\neq\phi \\ |x-y|=1}} (\phi_x - \phi_y)^2 \Big|_{\phi\equiv 0 \text{ on } \Lambda_N^c} = \frac{1}{2} \langle \phi, (-\Delta_N)\phi \rangle_{\Lambda_N},$$

Δ_N is a discrete Laplacian on $\mathbb{Z}^d$ with Dirichlet boundary condition outside Λ_N and $\langle \cdot, \cdot \rangle_{\Lambda_N}$ denotes $l^2(\Lambda_N)$-scalar product. $U : \mathbb{R} \to \mathbb{R}$ is a self-potential. The corresponding Gibbs measure is defined by

$$P_N^U(d\phi) = \frac{1}{Z_N^U} \exp\{-H_N^U(\phi)\} \prod_{x\in\Lambda_N} d\phi_x \prod_{x\notin\Lambda_N} \delta_0(d\phi_x), \quad (1)$$

where $d\phi_x$ denotes Lebesgue measure on $\mathbb{R}$, δ_0 denotes Dirac mass at 0 and Z_N^U is a normalization factor. We denote P_N the measure without the self-potential. In that case, the measure coincides with the law of a centered Gaussian field on $\mathbb{R}^{\Lambda_N}$ with the covariance matrix $(-\Delta_N)^{-1}$. It is well-known that the field has long range correlations under P_N and the following asymptotic behavior of the variance holds (cf. [9, Sect. 3], etc.):

$$\mathrm{Var}_{P_N}(\phi_0) = (-\Delta_N)^{-1}(0,0) \sim \begin{cases} g_2 \log N & \text{if } d = 2, \\ g_d & \text{if } d \geq 3, \end{cases} \quad (2)$$

as $N \to \infty$, where $g_2 = \frac{2}{\pi}$ and $g_d = (-\Delta)^{-1}(0,0)$ for $d \geq 3$. Δ is a discrete Laplacian on $\mathbb{Z}^d$. The configuration $\phi = \{\phi_x\}_{x\in\Lambda_N}$ is interpreted as an effective modelization of (discretized) phase separating random interface embedded in the $d+1$-dimensional space. The spin ϕ_x denotes the height of the interface at the position $x \in \Lambda_N$. Under P_N, the interface is said to be delocalized when $d = 2$ because the variance diverges as $N \to \infty$. While, when $d \geq 3$ the interface is localized because the variance remains finite. What we are interested in are the behavior of the interface under the measure P_N^U and to clarify the effects of various external potentials U. The earlier works have mainly focused on potentials of the following type. We briefly summarize the typical results. For detail, see [9, 14, 15] and references therein.

Pinning: This is the problem to study the effect of weak self-potential which attracts the interface to the height level 0. For example, when the potential U is given by $U(r) = -bI(|r| \leq a), a > 0, b > 0$, this is called square-well pinning potential. In this case, the field under P_N^U turns to be localized and massive for every $d \geq 2$ (cf. [2, 8]).

Entropic Repulsion: When U is (formally) given by $U(r) = \infty \cdot I(r < 0)$, this corresponds to the "hard wall" condition:

$$\Omega_N^+ = \{\phi; \phi_x \geq 0 \text{ for every } x \in \Lambda_N\},$$

and the measure P_N^U corresponds to the conditioned measure $P_N(\cdot\,|\Omega_N^+)$. The following result is known in this case (cf. [3, 4]):

$$\lim_{N\to\infty} \inf_{x\in\Lambda_{N,\varepsilon}} P_N\left(\left| \frac{1}{\sqrt{\log_d(N)}} \phi_x - \sqrt{4g_d} \right| \leq \delta |\Omega_N^+ \right) = 1, \quad (3)$$

for every $d \geq 2$, $\varepsilon > 0$ and $\delta > 0$ where $\log_2(N) = (\log N)^2, \log_d(N) = \log N$ for $d \geq 3$ and $\Lambda_{N,\varepsilon} = \{x \in \Lambda_N; \mathrm{dist}(x, \Lambda_N^c) \geq \varepsilon N\}$. Namely, the field is pushed up to the level

$\sqrt{4g_d}\sqrt{\log_d(N)}$. Since the estimate (2) means that the height of the interface $|\phi_0|$ is of order $O(\sqrt{\log N})$ when $d=2$ and $O(1)$ when $d\geq 3$ under the measure P_N, by (3) we see that once the hard wall is settled at height level 0, the interface is pushed up further at the order of $\sqrt{\log N}$ for every $d\geq 2$. Especially the interface turns to be delocalized when $d\geq 3$ by the hard wall. This phenomenon is called entropic repulsion and is caused by the random fluctuation of the interface that naturally arises from the Lebesgue measure $d\phi$ in the Gibbs measure (1), in other words, by entropic effects of the measure. The interface is shifted above to keep enough width of the fluctuation.

Also, the competition between pinning and hard-wall is related to the problem of *wetting transition* (cf. [15], etc.).

Now we are in the position to state the main result of this paper. The following theorem means that entropic repulsion for the massless field occurs even under quite weak repulsive force. We only require that the self-potential is non-increasing and the corresponding Gibbs measure is well-defined.

Theorem 1 *Let $d\geq 2$ and $U:\mathbb{R}\to\mathbb{R}$ be an arbitrary non-increasing, non-constant function which satisfies $Z_N^U<\infty$ for every $N\geq 1$. Then, for every $\varepsilon>0,\delta>0$ and $\gamma>0$, the following holds.*

$$\lim_{N\to\infty} P_N^U\left(\sharp\left\{x\in\Lambda_{N,\varepsilon};\phi_x\leq(1-\delta)\sqrt{4g_d}\sqrt{\log_d(N)}\right\}\geq\gamma|\Lambda_{N,\varepsilon}|\right)=0. \tag{4}$$

Also, if U satisfies the condition: there exists $a\in\mathbb{R}$ such that $U(r)=const.$ for every $r\geq a$, then we have

$$\lim_{N\to\infty} P_N^U\left(\sharp\left\{x\in\Lambda_{N,\varepsilon};\phi_x\geq(1+\delta)\sqrt{4g_d}\sqrt{\log_d(N)}\right\}\geq\gamma|\Lambda_{N,\varepsilon}|\right)=0. \tag{5}$$

In particular, a self-potential of the form $U(r)=bI(r<a), b>0$ pushes up the interface to the same level as the hard wall case and it does not depend on the parameter $b\in(0,\infty], a\in\mathbb{R}$. The effect of repulsive self-potentials was originally discussed in [13]. Our result gives a refinement of their result.

Remark 1 As we imposed in the theorem, some assumption on U is needed to obtain the upper bound (5) of the same level as (4). For example, consider the case that $U(r)=-\lambda r$, $\lambda>0$. Then random walk representation yields that $E^{P_N^U}[\phi_x]=\lambda\mathbb{E}_x[\tau_N]=O(N^2)$ as $N\to\infty$ where τ_N is the first exit time of simple random walk from Λ_N. Therefore the lower bound (4) is not optimal in this case.

The proof of Theorem 1 is given in the next section. Our strategy is as follows: by an FKG argument, essentially it is sufficient to prove the lower bound (4) for every non-increasing function which satisfies the following condition: there exist $a_-,a_+\in\mathbb{R}$ and $b>0$ such that $U(r)=0$ for $r<a_-$ and $U(r)=-b$ for $r\geq a_+$. For such a potential U, the corresponding Gibbs measure P_N^U can be represented as a weighted sum of a family of measures such that each measure is conditioned to be greater than a_- and has a certain self-potential on a subset of Λ_N (see (6) below). The weight of each measure gives a probability on $\mathcal{P}(\Lambda_N)$, the family of all subsets of Λ_N and it can be compared with a Bernoulli measure on $\{0,1\}^{\Lambda_N}$ in the FKG sense. Also, the conditioned measure with the self-potential is compared with the conditioned measure without the self-potential. As a result we can reduce our problem to the study of entropic repulsion above a rarefied wall, that is, a hard wall which locates only on the open vertices of Bernoulli site percolation on $\mathbb{Z}^d$. This problem is studied in Sect. 3.

Finally, we remark that throughout this paper below C represents a positive constant which does not depend on N but may depend on other parameters. Also, this C in estimates may change from place to place in the paper.

2 Proof of Theorem 1

2.1 Proof of the Lower Bound (4)

Let $U : \mathbb{R} \to \mathbb{R}$ be a non-increasing, non-constant function. Since adding a constant to the Hamiltonian does not affect the Gibbs measure, we may assume that there exist $a_-, a_+ \in \mathbb{R}$ such that $0 = U(a_-) > U(a_+)$. Set $U_0(r) := U(r)I_{[a_-,a_+)}(r) + U(a_+)I_{[a_+,\infty)}(r)$ and $U_1 := U - U_0$. Then U_0 and U_1 are non-increasing functions. Now, for every non-increasing function $f : \Omega_N \to \mathbb{R}$, we have

$$E^{P_N^U}[f] = \frac{Z_N^{U_0}}{Z_N^U} E^{P_N^{U_0}}\left[f \cdot \exp\left\{-\sum_{x\in\Lambda_N} U_1(\phi_x)\right\}\right]$$
$$\leq \frac{Z_N^{U_0}}{Z_N^U} E^{P_N^{U_0}}\left[f\right] E^{P_N^{U_0}}\left[\exp\left\{-\sum_{x\in\Lambda_N} U_1(\phi_x)\right\}\right] = E^{P_N^{U_0}}[f],$$

where the inequality follows from FKG inequality for $P_N^{U_0}$. Note that for an arbitrary self-potential U, FKG inequality holds for the measure P_N^U (cf. [10, Appendix B]). Therefore for the proof of the lower bound, it is sufficient to prove (4) for every non-increasing function U which satisfies the following condition: there exist $a_-, a_+ \in \mathbb{R}$ and $b > 0$ such that $U(r) = 0$ for $r < a_-$ and $U(r) = -b$ for $r \geq a_+$. From now on, we assume that U satisfies this condition.

By using the expansion

$$\exp\left\{-\sum_{x\in\Lambda_N} U(\phi_x)\right\} = \sum_{A\subset\Lambda_N}\prod_{x\in A}(e^{-U(\phi_x)} - 1)I(\phi_x \geq a_- \text{ for every } x \in A),$$

we have

$$E^{P_N^U}[f] = \sum_{A\subset\Lambda_N} \frac{\Xi_N^U(A)}{Z_N^U} E^{Q_N^{U,A}}[f], \tag{6}$$

for any bounded measurable function $f : \Omega_N \to \mathbb{R}$, where

$$Q_N^{U,A}(d\phi) = \frac{1}{\Xi_N^U(A)}\prod_{x\in A}(e^{-U(\phi_x)} - 1)I_{\Omega_{a_-}^+(A)}(\phi)e^{-H_N(\phi)}$$
$$\times \prod_{x\in\Lambda_N} d\phi_x \prod_{x\notin\Lambda_N}\delta_0(d\phi_x),$$
$$\Xi_N^U(A) = \int_{\mathbb{R}^{\Lambda_N}}\prod_{x\in A}(e^{-U(\phi_x)} - 1)I_{\Omega_{a_-}^+(A)}(\phi)e^{-H_N(\phi)}$$
$$\times \prod_{x\in\Lambda_N} d\phi_x \prod_{x\notin\Lambda_N}\delta_0(d\phi_x),$$

and $\Omega_a^+(A)$ denotes the event $\{\phi; \phi_x \geq a \text{ for every } x \in A\}$ for $a \in \mathbb{R}$, $A \subset \mathbb{Z}^d$.

Set $\rho_N(A) := \frac{\Xi_N^U(A)}{Z_N^U}$. Then $\sum_{A\subset\Lambda_N}\rho_N(A)=1$ and ρ_N gives a probability on $\mathcal{P}(\Lambda_N)$, the family of all subsets of Λ_N. We denote the corresponding random variable by $\widetilde{\mathcal{A}}_N$. Similarly to the pinning case (cf. [2]), we can compare this measure with a Bernoulli measure. Let $\sigma=\{\sigma_z\}_{z\in\mathbb{Z}^d}$ be i.i.d. $\{0,1\}$-valued random variables and ν_q, $q\in(0,1)$ denotes its law with $\nu_q(\sigma_z=1)=q=1-\nu_q(\sigma_z=0)$.

Lemma 1 *ρ_N is dominated by a Bernoulli measure on $\{0,1\}^{\Lambda_N}$ from above and below in the FKG sense. Namely, there exist $q_-=q_-(a_+,b)$, $q_+=q_+(b)\in(0,1)$ such that*

$$E^{\nu_{q-}}\big[g(\mathcal{A}_N)\big]\le E^{\rho_N}\big[g(\widetilde{\mathcal{A}}_N)\big]\le E^{\nu_{q+}}\big[g(\mathcal{A}_N)\big],$$

for every non-decreasing function $g:\mathcal{P}(\Lambda_N)\to\mathbb{R}$, where $\mathcal{A}_N=\mathcal{A}_N(\sigma)$ under the measure ν_q denotes the set $\{z\in\Lambda_N;\sigma_z=1\}$.

Proof For every $y\in\Lambda_N$ and $A\subset\Lambda_N\setminus\{y\}$, we have

$$\begin{aligned}\rho_N(y\in\widetilde{\mathcal{A}}_N|\widetilde{\mathcal{A}}_N\setminus\{y\}=A)&=\left(1+\frac{\rho_N(A)}{\rho_N(A\cup\{y\})}\right)^{-1}\\&=\left(1+\frac{\Xi_N^U(A)}{\Xi_N^U(A\cup\{y\})}\right)^{-1}.\end{aligned}$$

Now, set $h_A(\phi):=\prod_{x\in A}(e^{-U(\phi_x)}-1)\cdot I_{\Omega_{a_-}^+(A)}(\phi)$, $A\subset\Lambda_N$. Then

$$\begin{aligned}\frac{\Xi_N^U(A\cup\{y\})}{\Xi_N^U(A)}&=\frac{E^{P_N}[h_{A\cup\{y\}}]}{E^{P_N}[h_A]}\\&\ge E^{P_N}[h_{\{y\}}]\ge(e^b-1)P_N(\phi_y\ge a_+)\ge(e^b-1)C,\end{aligned}$$

for some constant $C=C(a_+)\in(0,1)$, where the first inequality follows from FKG inequality and the second inequality follows from the assumption that $U\equiv -b$ on $[a_+,\infty)$. Also, trivially we have $\frac{\Xi_N^U(A\cup\{y\})}{\Xi_N^U(A)}\le e^b-1$. Therefore, we obtain

$$\rho_N(y\in\widetilde{\mathcal{A}}_N|\widetilde{\mathcal{A}}_N\setminus\{y\}=A)\begin{cases}\ge\frac{(e^b-1)C}{(e^b-1)C+1}=:q_-,\\ \le\frac{e^b-1}{e^b}=:q_+,\end{cases}$$

for every $y\in\Lambda_N$ and $A\subset\Lambda_N\setminus\{y\}$. Hence the well-known Holley's criterion (cf. [11]) holds and we obtain the lemma. □

We also have the following lemma.

Lemma 2 *For every non-increasing function $f:\Omega_N\to\mathbb{R}$ and $A\subset\Lambda_N$, it holds that*

$$E^{Q_N^{U,A}}[f]\le E^{P_N}[f|\Omega_{a_-}^+(A)],$$

where the right hand side represents the expectation of the function f with respect to the conditioned measure $P_N(\,\cdot\,|\Omega_{a_-}^+(A))$.

Proof We can compute that

$$\begin{aligned}
&E^{Q_N^{U,A}}[f] \\
&\quad = \frac{Z_N P_N(\Omega_{a-}^+(A))}{\Xi_N^U(A)} E^{P_N}\Big[f \cdot \prod_{x\in A}(e^{-U(\phi_x)}-1)\Big|\Omega_{a-}^+(A)\Big] \\
&\quad \le \frac{Z_N P_N(\Omega_{a-}^+(A))}{\Xi_N^U(A)} E^{P_N}[f|\Omega_{a-}^+(A)]E^{P_N}\Big[\prod_{x\in A}(e^{-U(\phi_x)}-1)\Big|\Omega_{a-}^+(A)\Big] \\
&\quad = E^{P_N}[f|\Omega_{a-}^+(A)],
\end{aligned}$$

for every non-increasing function f, where the inequality follows from FKG inequality for the conditioned measure $P_N(\,\cdot\,|\Omega_{a-}^+(A))$. □

By Lemmas 1, 2 and (6), for every non-increasing function $f:\Omega_N\to\mathbb{R}$ and $\varepsilon>0$, we have

$$\begin{aligned}
E^{P_N^U}[f] &= E^{\rho_N}\big[E^{Q_N^{U,\widetilde{\mathcal{A}}_N}}[f]\big] \\
&\le E^{\rho_N}\big[E^{P_N}[f|\Omega_{a-}^+(\widetilde{\mathcal{A}}_N)]\big] \\
&\le E^{\nu_{q-}}\big[E^{P_N}[f|\Omega_{a-}^+(\mathcal{A}_N)]\big] \\
&\le E^{\nu_{q-}}\big[E^{P_N}[f|\Omega_{a-}^+(\mathcal{A}_{N,\varepsilon})]\big],
\end{aligned}$$

where $\mathcal{A}_{N,\varepsilon}=\{z\in\Lambda_{N,\varepsilon};\sigma_z=1\}$. Note that $E^{P_N}[f|\Omega_a^+(A)]$ is non-increasing in A if f is non-increasing. So the problem is reduced to the study of entropic repulsion above a rarefied wall, that is, a hard wall which locates only on the open vertices of Bernoulli site percolation on $\mathbb{Z}^d$ and the following proposition completes the proof of (4). The proof is given in the next section.

Proposition 1 *Let $d\ge 2$. For every $a\in\mathbb{R}$, $q\in(0,1)$, $0<\varepsilon'<\varepsilon<1$, $\delta>0$ and $\gamma>0$, the following holds.*

$$\begin{aligned}
\lim_{N\to\infty} E^{\nu_q}\Big[P_N\Big(\sharp\big\{x\in\Lambda_{N,\varepsilon};\phi_x\le(1-\delta)\sqrt{4g_d}\sqrt{\log_d(N)}\big\} \\
\ge\gamma|\Lambda_{N,\varepsilon}|\,\Big|\,\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\Big)\Big]=0.
\end{aligned}$$

Remark 2 The conclusion of this proposition was initially mentioned in p. 493 (3) of [1].

2.2 Proof of the Upper Bound (5)

Upper bound is easily given by comparison with the hard wall case. We may assume that there exists some $a\in\mathbb{R}$ such that $U(r)=0$ for every $r\ge a$. Set $U^{(k)}(r):=U(r)+k(a-r)^4I(r\le a)$, $k\ge 1$. Then $U^{(k)}-U$ is non-increasing continuous function and stochastic domination $P_N^U\prec P_N^{U^{(k)}}$ holds for every $k\ge 1$ (cf. [10, Appendix B]). Also $P_N^{U^{(k)}}$ weakly converges to $P_N(\,\cdot\,|\Omega_a^+(\Lambda_N))$ as $k\to\infty$. Therefore we have $E^{P_N^U}[f]\le E^{P_N}[f|\Omega_a^+(\Lambda_N)]$ for every non-decreasing function $f:\Omega_N\to\mathbb{R}$ and the upper bound follows from that for the hard wall case (cf. [3, 4]).

Remark 3 Since the proof of the upper bound relies on the result of the hard wall case, we can also obtain the upper bound of pointwise estimate (3) for P_N^U. On the other hand, we have not obtained the pointwise estimate of the lower bound. For the lower bound, the density estimate of high points like (4) and FKG argument yields the pointwise estimate (3) in the hard wall case (cf. [4, proof of (1.4)] etc.). But their FKG argument does not work well in our setting. For example, if $N \le M$, though stochastic domination $P_N(\,\cdot\,|\Omega_N^+) \prec P_M(\,\cdot\,|\Omega_M^+)$ holds, $P_N^U \prec P_M^U$ does not hold in general.

3 Entropic Repulsion Above a Rarefied Wall

In this section, we give the proof of Proposition 1. We will prove the quenched estimate:

$$\lim_{N\to\infty} P_N\Big(\sharp\big\{x \in \Lambda_{N,\varepsilon}; \phi_x \le (1-\delta)\sqrt{4g_d}\sqrt{\log_d(N)}\big\} \\ \ge \gamma|\Lambda_{N,\varepsilon}| \,\Big|\, \Omega_a^+(\mathcal{A}_{N,\varepsilon'})\Big) = 0,$$

for ν_q a.e. σ. For this purpose, we first characterize a typical good configuration of the random hard wall. Then, for fixed such a realization, we proceed the conditioning argument developed by [1, 3], and [5]. The basis of our proof is that by the long range correlation of the field, a hard wall at positive density points should be sufficient to push up the interface.

3.1 Proof of Proposition 1; The Higher Dimensional Case

We first prove the higher dimensional case $d \ge 3$. Let $a \in \mathbb{R}, 0 < \varepsilon' < \varepsilon < 1, \delta > 0, \gamma > 0, q \in (0,1)$ be fixed. We also choose $\eta > 0$ small enough and $L \in \mathbb{N}$ large enough. These are specified later on. For $z \in \Gamma_L := [0, 4L-1]^d \cap \mathbb{Z}^d$, set $\Lambda_{N,\varepsilon}^L(z) := (z + 4L\mathbb{Z}^d) \cap \Lambda_{N,\varepsilon}$. Since

$$\big\{\sharp\big\{x \in \Lambda_{N,\varepsilon}; \phi_x \le (1-\delta)\sqrt{4g_d}\sqrt{\log N}\big\} \ge \gamma|\Lambda_{N,\varepsilon}|\big\} \\ \subset \bigcup_{z\in\Gamma_L} \big\{\sharp\big\{x \in \Lambda_{N,\varepsilon}^L(z); \phi_x \le (1-\delta)\sqrt{4g_d}\sqrt{\log N}\big\} \ge \gamma|\Lambda_{N,\varepsilon}^L(z)|\big\},$$

it is sufficient to show that

$$\lim_{N\to\infty} E^{\nu_q}\Big[P_N\big(F_{\delta,\gamma}^0 |\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big)\Big] = 0,$$

where

$$F_{\delta,\gamma}^0 = \big\{\sharp\big\{x \in \Lambda_{N,\varepsilon}^L; \phi_x \le (1-\delta)\sqrt{4g_d}\sqrt{\log N}\big\} \ge \gamma|\Lambda_{N,\varepsilon}^L|\big\},$$

and

$$\Lambda_{N,\varepsilon}^L := \Lambda_{N,\varepsilon}^L(0).$$

At first, we consider the configuration of rarefied wall distributed by the Bernoulli measure ν_q. For $x \in 4L\mathbb{Z}^d$, set $\theta_x = I(\sigma_z = 1 \text{ for some } z \in B(x, \eta L))$ where $B(x,r) = \{y \in \mathbb{Z}^d; \max_{1\le i\le d} |y_i - x_i| \le r\}$ denotes a box on $\mathbb{Z}^d$ with centered at $x \in \mathbb{Z}^d$ and side length $2r+1$. Under ν_q, $\{\theta_x\}_{x\in 4L\mathbb{Z}^d}$ are i.i.d. and $E^{\nu_q}[\theta_x] = 1-(1-q)^{(2\eta L+1)^d}$. Take L large enough so that $(1-q)^{(2\eta L+1)^d} < \frac{1}{8}\gamma$. Then, large deviation estimate yields that

$$\begin{aligned}
&\nu_q\left(\sharp\left\{x\in\Lambda^L_{N,\varepsilon};\theta_x=0\right\}\ge\frac{1}{4}\gamma|\Lambda^L_{N,\varepsilon}|\right)\\
&\quad\le\nu_q\left(\frac{1}{|\Lambda^L_{N,\varepsilon}|}\sum_{x\in\Lambda^L_{N,\varepsilon}}(\theta_x-E^{\nu_q}[\theta_x])\le-\frac{1}{8}\gamma\right)\\
&\quad\le e^{-CN^d},
\end{aligned}$$

for every N large enough. Especially by Borel-Cantelli lemma, for ν_q a.e. σ, there exists some $N_0 = N_0(\sigma)\in\mathbb{N}$ such that $\sharp\{x\in\Lambda^L_{N,\varepsilon};\theta_x=1\}\ge(1-\frac{1}{4}\gamma)|\Lambda^L_{N,\varepsilon}|$ for every $N\ge N_0$. We fix such a realization σ and prove that $\lim_{N\to\infty}P_N(F^0_{\delta,\gamma}|\Omega^+_a(\mathcal{A}_{N,\varepsilon'}))=0$ for $\mathcal{A}_{N,\varepsilon'}=\mathcal{A}_{N,\varepsilon'}(\sigma)$.

For $x\in 4L\mathbb{Z}^d$ with $\theta_x=1$, we choose a point $z\in B(x,\eta L)$ such that $\sigma_z=1$ and denote it as $z=z(x)$. Set $\mathcal{D}_N := \{x\in\Lambda^L_{N,\varepsilon};\theta_x=1\}$ and $\widetilde{\mathcal{D}}_N := \{z(x);x\in\mathcal{D}_N\}$. We also define $m^L_x = E^{P_N}[\phi_x|\mathcal{F}^L]$, $v^L_x=\mathrm{Var}_{P_N}(\phi_x|\mathcal{F}^L)$ where $\mathcal{F}^L=\sigma(\phi_y;y\in\partial^+S(x,L),x\in\Lambda^L_{N,\varepsilon})$, $S(x,L)=\{y\in\mathbb{Z}^d;|y-x|\le L\}$ is a ball on $\mathbb{Z}^d$ with centered at x and radius L. For $A\subset\mathbb{Z}^d$, $\partial^+A=\{x\notin A;|x-y|=1\text{ for some }y\in A\}$ denotes its outer boundary. Now, consider the following events

$$\begin{aligned}
F^1_{\delta,\gamma}&=\left\{\sharp\left\{x\in\Lambda^L_{N,\varepsilon};m^L_x\le(1-\delta)\sqrt{4g_d}\sqrt{\log N}\right\}\ge\gamma|\Lambda^L_{N,\varepsilon}|\right\},\\
F^2_{\delta,\gamma}&=\left\{\sharp\left\{x\in\mathcal{D}_N;m^L_{z(x)}\le(1-\delta)\sqrt{4g_d}\sqrt{\log N}\right\}\ge\gamma|\mathcal{D}_N|\right\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&F^0_{\delta,\gamma}\cap\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\\
&\quad\subset\left(F^0_{\delta,\gamma}\cap(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma})^c\right)\cup\left(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma}\cap(F^2_{\frac{1}{4}\delta,\frac{1}{4}\gamma})^c\cap\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\right)\\
&\qquad\cup\left(F^2_{\frac{1}{4}\delta,\frac{1}{4}\gamma}\cap\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\right),
\end{aligned}$$

so

$$\begin{aligned}
&P_N\left(F^0_{\delta,\gamma}|\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\right)\\
&\quad\le P_N(\Omega^+_a(\mathcal{A}_{N,\varepsilon'}))^{-1}P_N\left(F^0_{\delta,\gamma}\cap(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma})^c\right)\\
&\qquad+P_N\left(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma}\cap(F^2_{\frac{1}{4}\delta,\frac{1}{4}\gamma})^c|\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\right)\\
&\qquad+P_N(\Omega^+_a(\mathcal{A}_{N,\varepsilon'}))^{-1}P_N\left(F^2_{\frac{1}{4}\delta,\frac{1}{4}\gamma}\cap\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\right)\\
&\quad=:I^1_N+I^2_N+I^3_N.
\end{aligned}$$

We show that each term in the right hand side goes to 0 as $N\to\infty$.

Estimate on I^1_N: On $F^0_{\delta,\gamma}\cap(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma})^c$, we have

$$\sharp\left\{x\in\Lambda^L_{N,\varepsilon};\phi_x-m^L_x\le-\frac{1}{2}\delta\sqrt{4g_d}\sqrt{\log N}\right\}\ge\frac{1}{2}\gamma|\Lambda^L_{N,\varepsilon}|.$$

Therefore,

$$P_N\big(F^0_{\delta,\gamma}\cap(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma})^c\big)\\ \le E^{P_N}\bigg[P_N\bigg(\frac{1}{|\Lambda^L_{N,\varepsilon}|}\sum_{x\in\Lambda^L_{N,\varepsilon}}I\bigg(\phi_x-m^L_x\le-\frac{1}{2}\delta\sqrt{4g_d}\sqrt{\log N}\bigg)\ge\frac{1}{2}\gamma\Big|\mathcal{F}^L\bigg)\bigg]. \quad (7)$$

Under the measure $P_N(\,\cdot\,|\mathcal{F}^L)$, $\{\phi_x-m^L_x\}_{x\in\Lambda^L_{N,\varepsilon}}$ are i.i.d. centered Gaussian random variables with the variance $v^L_x\le g_d$. Gaussian tail estimate shows that

$$E^{P_N}\bigg[I\bigg(\phi_x-m^L_x\le-\frac{1}{2}\delta\sqrt{4g_d}\sqrt{\log N}\bigg)\Big|\mathcal{F}^L\bigg]\le\exp\bigg\{-\frac{\delta^2}{2}\log N\bigg\}.$$

Hence by large deviation estimate, the right hand side of (7) is less than e^{-CN^d} for N large enough.

On the other hand, by the result of [4], we have

$$P_N(\Omega^+_a(\mathcal{A}_{N,\varepsilon'}))\ge P_N(\Omega^+_a(\Lambda_{N,\varepsilon'}))\ge e^{-CN^{d-2}\log N}.$$

Therefore we obtain $\lim_{N\to\infty}I^1_N=0$.

Estimate on I^2_N: At first, we have

$$F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma}\cap(F^2_{\frac{1}{4}\delta,\frac{1}{4}\gamma})^c\\ \subset\bigg\{\sharp\bigg\{x\in\mathcal{D}_N;\,m^L_{z(x)}-m^L_x\ge\frac{1}{4}\delta\sqrt{4g_d}\sqrt{\log N}\bigg\}\ge\frac{1}{16}\gamma^2|\Lambda^L_{N,\varepsilon}|\bigg\}\\ \subset\bigg\{\text{there exists some } x\in\mathcal{D}_N \text{ such that } m^L_{z(x)}-m^L_x\ge\frac{1}{4}\delta\sqrt{4g_d}\sqrt{\log N}\bigg\}.$$

Note that $|\mathcal{D}_N|\ge(1-\frac{1}{4}\gamma)|\Lambda^L_{N,\varepsilon}|$. Therefore, for the proof of $\lim_{N\to\infty}I^2_N=0$, it is sufficient to show that for $r:=\frac{1}{4}\delta\sqrt{4g_d}$ and L large enough it holds that

$$\sup_{x\in\Lambda^L_{N,\varepsilon}}\sup_{z\in B(x,\eta L)}P_N\big(m^L_z-m^L_x\ge r\sqrt{\log N}|\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\big)=o(N^{-d}), \quad (8)$$

as $N\to\infty$.

For $x\in\Lambda^L_{N,\varepsilon}$ and $z\in B(x,\eta L)$, set $\psi_{x,z}:=m^L_z-m^L_x$. For the proof of (8), we first estimate the moment $E^{P_N}[\psi_{x,z}|\Omega^+_a(\mathcal{A}_{N,\varepsilon'})]$. By random walk representation m^L_z can be represented as $m^L_z=\sum_{y\in\partial^+S(x,L)}H_L(z,y)\phi_y$ where $H_L(z,y)=\mathbb{P}_z(\xi_{\tau_{\partial^+S(x,L)}}=y)$, $\{\xi_n\}_{n\ge0}$ is a simple random walk on $\mathbb{Z}^d$, $\mathbb{P}_z$ denotes its law starting at z and $\tau_{\partial^+S(x,L)}$ is the first hitting time to $\partial^+S(x,L)$. Therefore,

$$E^{P_N}\big[\psi_{x,z}|\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\big]\\ \le\sum_{y\in\partial^+S(x,L)}|H_L(z,y)-H_L(x,y)|E^{P_N}\big[|\phi_y||\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\big].$$

By Lemma 4.1 of [1], there exists some constant $C_1>0$ such that, for every η small enough, we have $|H_L(z,y)-H_L(z',y)|\le C_1\eta L^{1-d}$ for every L large enough, $z,z'\in B(0,\eta L)$ and

$y \in \partial^+ S(0, L)$. Also, for every $y \in \Lambda_N$ we have

$$E^{P_N}\big[|\phi_y||\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big] = E^{P_N}\big[(\phi_y)^+|\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big] + E^{P_N}\big[(\phi_y)^-|\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big],$$

where we denote $t^+ := \max\{t, 0\}$, $t^- := \max\{-t, 0\}$ for $t \in \mathbb{R}$. By (3), we estimate that

$$E^{P_N}\big[(\phi_y)^+|\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big] \le E^{P_N}\big[(\phi_y)^+|\Omega_a^+(\Lambda_N)\big] \le C\sqrt{\log N},$$

and by (2),

$$E^{P_N}\big[(\phi_y)^-|\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big] \le E^{P_N}\big[(\phi_y)^-\big] \le E^{P_N}\big[((\phi_y)^-)^2\big]^{\frac{1}{2}} = \left(\frac{1}{2}E^{P_N}\big[(\phi_y)^2\big]\right)^{\frac{1}{2}} \le C.$$

Recall that $E^{P_N}[f|\Omega_a^+(A)]$ is non-increasing in A if f is non-increasing. Hence we have

$$\sup_{y\in\Lambda_N} E^{P_N}\big[|\phi_y||\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big] \le C\sqrt{\log N},$$

for every N large enough. By these estimates, there exists some constant $C_2 > 0$ such that for every $\eta > 0$ small enough we have

$$E^{P_N}\big[\psi_{x,z}|\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big] \le C_2\eta\sqrt{\log N},$$

for every N large enough.

Now, if $C_2\eta < \frac{1}{2}r$ then

$$\begin{aligned}
&P_N\big(\psi_{x,z} \ge r\sqrt{\log N}|\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big)\\
&\quad\le P_N\left(|\psi_{x,z} - E^{P_N}[\psi_{x,z}|\Omega_a^+(\mathcal{A}_{N,\varepsilon'})]| \ge \frac{1}{2}r\sqrt{\log N}|\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\right)\\
&\quad\le \exp\left\{-\frac{1}{2\mathrm{Var}_{P_N}(\psi_{x,z})}\frac{1}{4}r^2\log N\right\},
\end{aligned} \tag{9}$$

where the second inequality follows from Brascamp-Lieb inequality for the measure $P_N(\,\cdot\,|\Omega_a^+(\mathcal{A}_{N,\varepsilon'}))$ (cf. [7, Appendix]). Note that $\psi_{x,z}$ is represented as a linear sum of $\{\phi_y\}_{y\in\partial^+S(x,L)}$. Also, we have the following estimate on the variance of $\psi_{x,z}$.

Lemma 3 *There exists some constant $C_3 > 0$ such that for every $\eta > 0$ small enough, $x \in \Lambda_{N,\varepsilon}^L$ and $z \in B(x, \eta L)$ it holds that*

$$\mathrm{Var}_{P_N}(\psi_{x,z}) = \mathrm{Var}_{P_N}(m_z^L - m_x^L) \le C_3\eta.$$

Proof The same statement for the case $d = 2$ has been proved in Lemma 12 of [3]. Since the proof of the case $d \ge 3$ is almost the same, we omit the detail. We have only to use the following estimate on the Green function of the simple random walk in their argument.

$$\mathbb{E}_0\left[\sum_{n=0}^{\infty}(I(\xi_n = x + y) - I(\xi_n = x))\right] = g_d|y|D_u(|x|^{2-d}) + O(|x|^{-d}),$$

for every $y \in \mathbb{Z}^d$, $y = |y|u$ (cf. [12, Theorem 1.5.5]). □

By (9) and Lemma 3, we obtain

$$P_N\big(\psi_{x,z} \geq r\sqrt{\log N}|\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big) \leq N^{-\frac{r^2}{8C_4\eta}}.$$

Therefore, if η satisfies $\frac{r^2}{8C_4\eta} > d$ then (8) holds.

Estimate on I_N^3: For I_N^3, we have that

$$P_N\big(F^2_{\frac{1}{4}\delta,\frac{1}{4}\gamma} \cap \Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big) \leq E^{P_N}\Bigg[\prod_{z\in\widetilde{\mathcal{D}}_N} P_N(\phi_z \geq a|\mathcal{F}^L);\, F^2_{\frac{1}{4}\delta,\frac{1}{4}\gamma}\Bigg].$$

For given $\mathcal{F}^L$, if $m_z^L \leq (1-\frac{1}{4}\delta)\sqrt{4g_d}\sqrt{\log N}$ for $z \in \widetilde{\mathcal{D}}_N$ then

$$\begin{aligned} P_N(\phi_z \geq a|\mathcal{F}^L) &\leq 1 - P_N\bigg(\phi_z - m_z^L < -\Big(1-\frac{1}{5}\delta\Big)\sqrt{4g_d}\sqrt{\log N}\Big|\mathcal{F}^L\bigg) \\ &\leq 1 - \frac{1}{Cv_z^L}\exp\bigg\{-\frac{1}{2v_z^L}\Big(1-\frac{1}{5}\delta\Big)^2 4g_d\log N\bigg\}, \end{aligned} \tag{10}$$

for every N large enough. Since $\mathrm{dist}(z, \partial^+S(x,L)) \geq CL$ for every $z \in B(x,\eta L)$ and $\lim_{L\to\infty} v_z^L = g_d$ for fixed z, we have $\inf_{z\in B(x,\eta L)} v_z^L > (1-\frac{1}{5}\delta)^2 g_d$ for L large enough. Hence, there exist some $C > 0$ and $\beta > 0$ such that the right hand side of (10) is less than $1 - CN^{-2+\beta}$ in this case. Since the number of such $z \in \widetilde{\mathcal{D}}_N$ is greater than $\frac{1}{4}\gamma|\mathcal{D}_N|$ on $F^2_{\frac{1}{4}\delta,\frac{1}{4}\gamma}$ and $|\mathcal{D}_N| \geq (1-\frac{1}{4}\gamma)|\Lambda^L_{N,\varepsilon}|$, we obtain $P_N(F^2_{\frac{1}{4}\delta,\frac{1}{4}\gamma} \cap \Omega_a^+(\mathcal{A}_{N,\varepsilon'})) \leq e^{-CN^{d-2+\beta}}$ for every N large enough. By combining this estimate with the lower bound of $P_N(\Omega_a^+(\mathcal{A}_{N,\varepsilon'}))$, we obtain $\lim_{N\to\infty} I_N^3 = 0$.

In conclusion, choose $\eta > 0$ small enough first to satisfy all the conditions above and choose L large enough accordingly. Then, by proceeding all the argument from the beginning, we can obtain the desired result.

3.2 Proof of Proposition 1; The Two Dimensional Case

Next, we consider the case $d = 2$. Let $a \in \mathbb{R}, 0 < \varepsilon' < \varepsilon < 1, \delta > 0, \gamma > 0, q \in (0,1)$ be fixed. We also choose $\alpha, \beta, \eta \in (0,1)$. α is close to 1 and β, η are close to 0. These are specified later on. Set $\Lambda^\alpha_{N,\varepsilon}(z) := (z + 4N^\alpha\mathbb{Z}^2) \cap \Lambda_{N,\varepsilon}, z \in \Gamma^\alpha_N := [0, 4N^\alpha - 1]^2 \cap \mathbb{Z}^2$. Since

$$\begin{aligned} &\big\{\sharp\big\{x \in \Lambda_{N,\varepsilon}; \phi_x \leq (1-\delta)\sqrt{4g_2}\log N\big\} \geq \gamma|\Lambda_{N,\varepsilon}|\big\} \\ &\quad\subset \bigcup_{z\in\Gamma^\alpha_N}\big\{\sharp\big\{x \in \Lambda^\alpha_{N,\varepsilon}(z); \phi_x \leq (1-\delta)\sqrt{4g_2}\log N\big\} \geq \gamma|\Lambda^\alpha_{N,\varepsilon}(z)|\big\}, \end{aligned}$$

it is sufficient to show that

$$E^{\nu_q}\big[P_N(F^0_{\delta,\gamma}|\Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big] = o(N^{-2\alpha}),$$

as $N \to \infty$, where

$$F^0_{\delta,\gamma} = \big\{\sharp\big\{x \in \Lambda^\alpha_{N,\varepsilon}; \phi_x \leq (1-\delta)\sqrt{4g_2}\log N\big\} \geq \gamma|\Lambda^\alpha_{N,\varepsilon}|\big\},$$

and $\Lambda^\alpha_{N,\varepsilon} := \Lambda^\alpha_{N,\varepsilon}(0)$.

At first, we consider a partition of $\Lambda_{N,\varepsilon}$ into boxes with side-length $2N^{\beta}+1$. The boundaries of neighboring boxes intersect and for simplicity, we always assume that $\Lambda_{N,\varepsilon}$ and every boxes with centered at $\Lambda^{\alpha}_{N,\varepsilon}$, side-length $2N^{\alpha}+1$ can be divided into such boxes without reminder. We denote by Π^{β}_{N} the set of these boxes in $\Lambda_{N,\varepsilon}$. For every $B\in\Pi^{\beta}_{N}$, large deviation estimate yields that

$$\nu_q\left(\sharp\{x\in B;\sigma_x=1\}\le\frac{1}{2}q|B|\right)\le e^{-C|B|}.$$

Hence we have

$$\begin{aligned}&\nu_q\left(\text{there exists some } B\in\Pi^{\beta}_{N} \text{ such that } \sharp\{x\in B;\sigma_x=1\}\le\frac{1}{2}q|B|\right)\\&\quad\le e^{-CN^{2\beta}},\end{aligned}$$

for every N large enough. Especially by Borel-Cantelli lemma, for ν_q-a.e. σ, there exists some $N_0=N_0(\sigma)\in\mathbb{N}$ such that for every $N\ge N_0$ it holds that $\sharp\{x\in B;\sigma_x=1\}\ge\frac{1}{2}q|B|$ for every $B\in\Pi^{\beta}_{N}$. We fix such a realization σ and prove that $P_N(F^0_{\delta,\gamma}|\Omega^+_a(\mathcal{A}_{N,\varepsilon'}))=o(N^{-2\alpha})$ as $N\to\infty$ for $\mathcal{A}_{N,\varepsilon'}=\mathcal{A}_{N,\varepsilon'}(\sigma)$.

Now, consider the following events

$$\begin{aligned}F^1_{\delta,\gamma}&=\left\{\sharp\{x\in\Lambda^{\alpha}_{N,\varepsilon};m^{\alpha}_x\le(1-\delta)\sqrt{4g_2}\log N\}\ge\gamma|\Lambda^{\alpha}_{N,\varepsilon}|\right\},\\F^2_{\delta}&=\Big\{\text{there exist some } x\in\Lambda^{\alpha}_{N,\varepsilon} \text{ and } y\in B(x,\eta N^{\alpha})\\&\qquad\text{such that } |m^{\alpha}_x-m^{\alpha}_y|\ge\delta\sqrt{4g_2}\log N\Big\},\end{aligned}$$

where we set $m^{\alpha}_x=E^{P_N}[\phi_x|\mathcal{G}^{\alpha}_N]$, $v^{\alpha}_x=\mathrm{Var}_{P_N}(\phi_x|\mathcal{G}^{\alpha}_N)$ and $\mathcal{G}^{\alpha}_N=\sigma(\phi_y;y\in\partial^+B(x,N^{\alpha}),x\in\Lambda^{\alpha}_{N,\varepsilon})$. Then we have

$$F^0_{\delta,\gamma}\cap\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\subset\left(F^0_{\delta,\gamma}\cap(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma})^c\right)\cup F^2_{\frac{1}{4}\delta}\cup\left(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma}\cap(F^2_{\frac{1}{4}\delta})^c\cap\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\right),$$

and

$$\begin{aligned}P_N\left(F^0_{\delta,\gamma}|\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\right)&\le P_N(\Omega^+_a(\mathcal{A}_{N,\varepsilon'}))^{-1}\Big\{P_N\left(F^0_{\delta,\gamma}\cap(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma})^c\right)+P_N\left(F^2_{\frac{1}{4}\delta}\right)\\&\qquad+P_N\left(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma}\cap(F^2_{\frac{1}{4}\delta})^c\cap\Omega^+_a(\mathcal{A}_{N,\varepsilon'})\right)\Big\}\\&=:I^1_N+I^2_N+I^3_N.\end{aligned}$$

We estimate each term in the right hand side.

Estimate on I^1_N: I^1_N can be estimated in the similar manner to the higher dimensional case. We have that

$$\begin{aligned}&P_N\left(F^0_{\delta,\gamma}\cap(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma})^c\right)\\&\quad\le E^{P_N}\left[P_N\left(\frac{1}{|\Lambda^{\alpha}_{N,\varepsilon}|}\sum_{x\in\Lambda^{\alpha}_{N,\varepsilon}}I\left(\phi_x-m^{\alpha}_x\le-\frac{1}{2}\delta\sqrt{4g_2}\log N\right)\ge\frac{1}{2}\gamma\Big|\mathcal{G}^{\alpha}_N\right)\right].\qquad(11)\end{aligned}$$

Under the measure $P_N(\,\cdot\,|\mathcal{G}_N^\alpha)$, $\{\phi_x - m_x^\alpha\}_{x\in\Lambda_{N,\varepsilon}^\alpha}$ are i.i.d. centered Gaussian random variables with the variance $v_x^\alpha = g_2 \log N^\alpha(1+o(1))$ as $N\to\infty$ and Gaussian tail estimate shows that

$$E^{P_N}\left[I\left(\phi_x - m_x^\alpha \le -\frac{1}{2}\delta\sqrt{4g_2}\log N\right)\Big|\mathcal{G}_N^\alpha\right] \le \exp\left\{-\frac{\delta^2}{2\alpha}\log N\right\}.$$

Hence by large deviation estimate, the right hand side of (11) is less than $e^{-C|\Lambda_{N,\varepsilon}^\alpha|} = e^{-CN^{2-2\alpha}}$ for N large enough.

On the other hand, by the result of [3], we have

$$P_N(\Omega_a^+(\mathcal{A}_{N,\varepsilon'})) \ge P_N(\Omega_a^+(\Lambda_{N,\varepsilon'})) \ge e^{-C(\log N)^2}.$$

Therefore we obtain that $I_N^1 = o(N^{-2\alpha})$ as $N\to\infty$.

Estimate on I_N^2: For I_N^2, we have

$$P_N\big(F^2_{\frac{1}{4}\delta}\big) \le CN^2 \sup_{x\in\Lambda_{N,\varepsilon}^\alpha}\ \sup_{y\in B(x,\eta N^\alpha)} P_N\left(|m_x^\alpha - m_y^\alpha| \ge \frac{1}{4}\delta\sqrt{4g_2}\log N\right).$$

For $x\in\Lambda_{N,\varepsilon}^\alpha$ and $y\in B(x,\eta N^\alpha)$, $m_x^\alpha - m_y^\alpha$ under P_N is a centered Gaussian random variable and by Lemma 12 of [3], $\mathrm{Var}_{P_N}(m_x^\alpha - m_y^\alpha) \le C_1\eta$ for some $C_1>0$ if $\eta>0$ is small enough. Therefore Gaussian tail estimate yields that $P_N(F^2_{\frac{1}{4}\delta}) \le e^{-\frac{C_2}{\eta}(\log N)^2}$ for every N large enough. Combining this estimate with the lower bound of $P_N(\Omega_a^+(\mathcal{A}_{N,\varepsilon'}))$, we obtain that $I_N^2 = o(N^{-2\alpha})$ if η is small enough.

Estimate on I_N^3: For I_N^3, we have

$$\begin{aligned}
&F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma} \cap (F^2_{\frac{1}{4}\delta})^c \cap \Omega_a^+(\mathcal{A}_{N,\varepsilon'})\\
&\quad\subset \left\{\sharp\left\{x\in\Lambda_{N,\varepsilon}^\alpha; m_y^\alpha \le \left(1-\frac{1}{4}\delta\right)\sqrt{4g_2}\log N \text{ for every } y\in B(x,\eta N^\alpha)\right\}\right.\\
&\qquad\left.\ge \frac{1}{2}\gamma|\Lambda_{N,\varepsilon}^\alpha|\right\} \cap \Omega_a^+(\mathcal{A}_{N,\varepsilon'})\\
&\quad\subset \left\{\sharp\left\{x\in\Lambda_{N,\varepsilon}^\alpha; \phi_y - m_y^\alpha \ge -\left(1-\frac{1}{5}\delta\right)\sqrt{4g_2}\log N\right.\right.\\
&\qquad\left.\left.\text{for every } y\in B(x,\eta N^\alpha)\cap\mathcal{A}_{N,\varepsilon'}\right\} \ge \frac{1}{2}\gamma|\Lambda_{N,\varepsilon}^\alpha|\right\},
\end{aligned}$$

for every N large enough. Therefore,

$$\begin{aligned}
&P_N\big(F^1_{\frac{1}{2}\delta,\frac{1}{2}\gamma} \cap (F^2_{\frac{1}{4}\delta})^c \cap \Omega_a^+(\mathcal{A}_{N,\varepsilon'})\big)\\
&\quad\le E^{P_N}\left[P_N\left(\frac{1}{|\Lambda_{N,\varepsilon}^\alpha|}\sum_{x\in\Lambda_{N,\varepsilon}^\alpha}\zeta_x \ge \frac{1}{2}\gamma\Big|\mathcal{G}_N^\alpha\right)\right],
\end{aligned} \tag{12}$$

where

$$\zeta_x := I\left(\phi_y - m_y^\alpha \ge -\left(1-\frac{1}{5}\delta\right)\sqrt{4g_2}\log N \text{ for every } y\in B(x,\eta N^\alpha)\cap\mathcal{A}_{N,\varepsilon'}\right).$$

Under the measure $P_N(\,\cdot\,|\mathcal{G}_N^\alpha)$, $\{\zeta_x\}_{x\in\Lambda_{N,\varepsilon}^\alpha}$ are independent $\{0,1\}$-valued random variables. Therefore, if we have

$$\lim_{N\to\infty}\sup_{x\in\Lambda_{N,\varepsilon}^\alpha} E^{P_N}[\zeta_x|\mathcal{G}_N^\alpha]=0, \tag{13}$$

then we can apply a concentration estimate (e.g. [6, Corollary 2.4.14]) and the right hand side of (12) is less than $e^{-C|\Lambda_{N,\varepsilon}^\alpha|}=e^{-CN^{2-2\alpha}}$. Hence we obtain that $I_N^3=o(N^{-2\alpha})$.

Finally, for the proof of (13), we prepare the following lemma.

Lemma 4 *Let $d=2$. Assume that $A\subset\mathbb{Z}^2$ satisfies the following condition: there exist $\beta_0\in(0,1)$ small enough and $C_0\in(0,1)$ such that for every N large enough, it holds that $|B\cap A|\ge C_0|B|$ for every $B\in\Pi_N^{\beta_0}$. Then, for every $\varepsilon>0$ and $r<1$, it holds that*

$$\lim_{N\to\infty} P_N\big(\phi_x\ge -r\sqrt{4g_2}\log N \text{ for every } x\in\Lambda_{N,\varepsilon}\cap A\big)=0.$$

The proof of this lemma is given in the end of this section. Now, for every $x\in\Lambda_{N,\varepsilon}^\alpha$, $\{\phi_y-m_y^\alpha\}_{y\in B(x,N^\alpha)}$ under $P_N(\,\cdot\,|\mathcal{G}_N^\alpha)$ is a centered Gaussian field with the covariance matrix $(-\Delta_{N^\alpha})^{-1}$. Therefore, by taking $\alpha,\beta\in(0,1)$ as $\frac{1}{\alpha}(1-\frac{1}{5}\delta)<1$ and $\frac{\beta}{\alpha}=\beta_0$, we can apply Lemma 4 and obtain (13). Note that $|B\cap\mathcal{A}_{N,\varepsilon'}|\ge\frac{1}{2}q|B|$ for every $B\in\Pi_N^\beta\cap B(x,\eta N^\alpha)$ by the choice of σ.

Proof of Lemma 4 The proof is given by a multi-scale analysis and is the same as the proof of [5, Theorem 1.3]. Roughly speaking, if the region A spreads evenly over $\mathbb{Z}^2$ and has positive densities in every boxes with small scale, then the event that the height of the field is greater than $-r\sqrt{4g_2}\log N$, $r<1$ for every points in $\Lambda_{N,\varepsilon}\cap A$ is incompatible with the fact that the 2-dimensional massless field has spikes of height $\alpha\sqrt{4g_2}\log N$ in the region of the length scale $O(N^\alpha)$, $0<\alpha<1$ (cf. [3, 5]).

We introduce some notations. Set $\Pi_N^0=\{\{x\};x\in A\cap\Lambda_{N,\varepsilon}\}$ and for $0<\alpha<1$, let Π_N^α be the collection of adjacent sub-boxes of side length $2N^\alpha+1$ in $\Lambda_{N,\varepsilon}$ such that $\Lambda_{N,\varepsilon}=\bigcup_{B\in\Pi_N^\alpha}B$. For simplicity we assume that $\Lambda_{N,\varepsilon}$ can be divided into such boxes without reminder. Next, we define a collection of boxes as follows. Fix $\frac{1}{2}<\alpha<1$, an integer $K\ge 2$ and let $\alpha_i:=\frac{K-i+1}{K}\alpha$, $1\le i\le K+1$. We first set $\Gamma_{\alpha_1}:=\Pi_N^{\alpha_1}$. Then assuming that Γ_{α_i} has been defined, for any $B\in\Gamma_{\alpha_i}$ we draw a square of side length $N^{\alpha_i}+1$ with the same center as B. The collection of sub-boxes in $\Pi_N^{\alpha_{i+1}}$ that intersect that square is called $\Gamma_{B,\alpha_{i+1}}$ and let $\Gamma_{\alpha_{i+1}}:=\bigcup_{B\in\Gamma_{\alpha_i}}\Gamma_{B,\alpha_{i+1}}$. For any box B, we write x_B for the center of B and $\phi_B:=E^{P_N}[\phi_{x_B}|\mathcal{F}_{\partial B}]$ where $\mathcal{F}_A=\sigma(\phi_x;x\in A)$, $A\subset\mathbb{Z}^d$. Next for $0<\eta<1$ and $2\le k\le K+1$, we set

$$n_k:=N^{\kappa+\frac{2\alpha(k-1)}{K}(1-\eta^2)}$$

and

$$\begin{aligned} D_k &:= \big\{\underline{B}^{(k)};\phi_{B_i}\ge(\alpha-\alpha_i)\eta(1-\gamma_K)\sqrt{4g_2}\log N \text{ for every } 1\le i\le K\big\},\\ C_k &:= \big\{\sharp D_k\ge n_k\big\}, \end{aligned}$$

where $\kappa>0$ is a constant in the proof of [5, Theorem 1.3], $\gamma_K:=\frac{1}{K}$ and $\underline{B}^{(k)}$ denotes a sequence of boxes $(B_1,\dots,B_k)$ which satisfies $B_1\supset B_2\supset\cdots\supset B_k$ and $B_i\in\Gamma_{\alpha_i}$, $1\le i\le k$. Actually, the difference to the proof of the lower bound of [5, Theorem 1.3] is only the definition of Π_N^0. The definition of $\Gamma_{B,\alpha_{K+1}}$ and $\Gamma_{\alpha_{K+1}}$ changes accordingly.

Now take $\alpha < 1, \eta < 1$ close to 1 and K large enough so that $\alpha\eta(1-\gamma_K) > r$. Then we have

$$\begin{aligned}&\left\{\phi_x \le r\sqrt{4g_2}\log N \text{ for every } x \in \Lambda_{N,\varepsilon} \cap A\right\}\\ &\subset \left\{\sharp\left\{x \in \Lambda_{N,\varepsilon} \cap A; \phi_x \ge \alpha\eta(1-\gamma_K)\sqrt{4g_2}\log N\right\} < n_{K+1}\right\}\\ &\subset C_{K+1}^c.\end{aligned}$$

Therefore, by symmetry it is sufficient to prove $\lim_{N\to\infty} P_N(C_{K+1}^c) = 0$. This can be proved by the completely same argument to the proof of the lower bound of [5, Theorem 1.3]. In fact, the difference is only the estimates which concern with $|\Gamma_{B,\alpha_{K+1}}|$, $B \in \Gamma_{\alpha_K}$ and if β_0 is less than the parameter $\alpha_K = \frac{\alpha}{K}$ (N^{α_K} is the smallest mesoscopic scale in the multi-scale analysis), then by our assumption it holds that $|\Gamma_{B,\alpha_{K+1}}| \ge \frac{1}{4}C_0|B| \ge C_0 N^{\frac{2\alpha}{K}}$ for every box $B \in \Gamma_{\alpha_K}$. This estimate is the same as the one in [5] and all the arguments there work well. □

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